Indirect control and power in mutual control structures

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Abstract

In a mutual control structure agents exercise control on each other. Typical examples can be found in corporate governance: firms and investment companies exercise mutual control, in particular by owning each others’ stocks. In this paper we formulate a general model for such situations. There is a fixed set of agents, and a mutual control structure assigns to each subset (coalition) the subset of agents controlled by that coalition. Such a mutual control structure captures direct control. We propose a procedure in order to incorporate indirect control as well: if $S$ controls $T$, and $S$ and $T$ jointly control $R$, then $S$ controls $R$ indirectly. This way, invariant mutual control structures result. Alternatively, mutual control can be described by vectors of simple games, each simple game describing who controls a certain player, and also those simple games can be updated in order to capture indirect control. We show that both approaches lead to equivalent invariant structures.

In the second part of the paper, we axiomatically develop a class of power indices for invariant mutual control structures. The four axioms we impose have a natural interpretation in this framework, and together they characterize a broad class of power indices based on dividends resulting both from exercising and from undergoing control. By adding an extra condition a unique power index is singled out. In this index, each player accumulates his Shapley-Shubik power index assignments from controlling other players, diminished by the joint Shapley-Shubik power index assignments to other players controlling him.

Keywords: Mutual control structure, simple games, power index.
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1 Introduction

A mutual control structure refers to a situation in which agents exercise control on each other. Typical examples occur in the area of corporate governance: there is a conglomerate of firms and investment companies who control each other, specifically by possessing shares or stocks. In such situations the question arises who are, ultimately, in control, and how much power the different agents have. See Crama and Leruth (2011) for a recent overview of both theoretical and empirical literature in this area. The purpose of the present paper is to answer these questions and add to the literature by developing a general game-theoretic model.

Formally, a mutual control structure $C$ will be a map assigning to each nonempty coalition – i.e., a subset of a given finite set of players $N$ – another coalition. The interpretation of $C(S) = T$ is, that each player of $T$ is controlled by the coalition $S$. For instance, $i \in T$ is a firm, and the coalition $S$ of firms or investment companies has a majority of the shares of firm $i$. We impose the natural condition of monotonicity: if $S$ controls $T$, then any coalition containing $S$ also controls $T$. While the mutual control structure $C$ thus captures direct control, it does not necessarily capture indirect control. The latter means that whenever $S$ controls $T$ and $S$ and $T$ jointly control $R$, then $S$ indirectly controls $R$. Thus, formally, one would like to have that, if $C(S) = T$ and $C(S \cup T) = R$, then also $C(S) = R$. In our example, if $j$ is a firm in $R$ and $S$ and $T$ jointly have a majority of the shares of $j$, then $S$ controls $j$ since $S$ controls all firms in $T$. A mutual control structure will be called invariant if it satisfies this condition. In the paper we start out by studying an algorithm which assigns to each mutual control structure its unique minimal invariant extension.

Alternatively, a mutual control structure can be described by a vector of simple games. For each player, there is a simple game with as winning coalitions exactly those coalitions which control that player. There is a one-to-one correspondence between mutual control structures and vectors of monotonic simple games. We propose an updating procedure, in which players can be replaced by coalitions which are winning in the simple game describing who controls that player. We show that this updating procedure results in a unique minimal invariant extension of the vector of simple games, and that the associated mutual control structure is indeed the minimal invariant extension of the originally given mutual control structure.

The approach in the literature which seems to be closest related to ours, is Gambarelli and Owen (1994). This approach explicitly distinguishes between firms and investors. In what they call a reduction, all power is reduced to power of the investors, i.e., the firms leave the scene. The proposed reduction operation bears some resemblance to our algorithm of making a mutual control structure invariant. Gambarelli and Owen (1994) end up with so-called consistent reductions which, however, are not necessarily unique, in contrast to our minimal invariant extensions. There are some other approaches in the literature aiming at establishing indirect control relations: see for instance Crama and Leruth (2007) and the already mentioned survey Crama and Leruth (2011).
In the second part of the paper we consider invariant mutual control structures and develop a class of power indices, intended to capture the ‘true’ power of individual players. We impose four axioms which have a natural interpretation in the present framework. First, we set the power of null-players equal to zero – a null-player is a player who neither contributes to controlling any other player, nor is controlled by any other player or coalition himself. Second, we scale the power index such that the sum of all assigned powers is always equal to zero. Third, we impose anonymity: the names of the players should not matter. Fourth, we impose a so-called transfer property, which says the following. For every player, the change in power when extending a mutual control structure $C'$ to $C$ should be equal to the change in power when extending a mutual control structure $D'$ to $D$, whenever exactly the same control relations are added going from $C'$ to $C$ as when going from $D'$ to $D$. This condition is called transfer property because it is related to the transfer property used by Dubey (1975) to characterize the Shapley value or Shapley-Shubik index (Shapley, 1953; Shapley and Shubik, 1954) for monotonic simple games. See also Dubey et al (2005).

We characterize the class of power indices satisfying these four conditions. Each power index in this class corresponds to a weight vector of dimension $2n-2$ (where $n$ is the number of players) and assigns to a player $i$ a weighted sum of dividends obtained in the simple games capturing the control undergone by the other players, diminished by dividends gathered by the other players in the game describing the control undergone by player $i$. By adding a natural monotonicity condition we restrict this class to power indices associated with nonnegative weight vectors. By imposing more scaling we obtain a unique power index with all weights equal to 1. This means that each player $i$ obtains the sum of all his Shapley-Shubik values in the games in which he contributes to controlling the other players, minus the sum of all Shapley-Shubik values of the other players in the game describing the control undergone by $i$.

To conclude, our approach can be applied whenever elementary, direct control relations can be established – for instance by considering simple majority share holdings within a corporate structure. Next, indirect control relations can be determined, and to the resulting invariant mutual control structure a power index can be applied.

In Section 2 we introduce and study mutual control structures, in Section 3 we consider the approach by simple games, and in Section 4 we develop our class of power indices. Section 6 concludes.

**Notation** For a set $A$ we denote by $P(A)$ the set of all subsets of $A$. By $|A|$ we denote the number of elements of $A$.

## 2 Mutual control structures

Let $N = \{1, \ldots, n\}$ with $n \geq 2$ denote the set of players. By $P(N)$ we denote the set of all subsets of $N$. Elements of $P(N)$ are called coalitions.
Definition 2.1. A mutual control structure (mcs) is a map \( C : P(N) \to P(N) \) satisfying

(i) \( C(\emptyset) = \emptyset \),

(ii) monotonicity: \( C(S) \subseteq C(T) \) for all \( S,T \in P(N) \) with \( S \subseteq T \).

The set of all mutual control structures is denoted by \( C \).

If \( i \in C(S) \) for some \( i \in N \) and \( S \in P(N) \) then we say that player \( i \) is controlled by coalition \( S \). (For instance, the firms in \( S \) hold a majority of the shares of firm \( i \).) Similarly, we say that \( S \) controls \( C(S) \). Thus, the empty coalition controls no one, and a player controlled by a coalition \( S \) is also controlled by any coalition \( T \) containing \( S \).

A mutual control structure does not necessarily capture ‘indirect’ control: if \( S \) controls \( T \), and \( S \) and \( T \) together control \( R \), then for an arbitrary mutual control structure it is not necessarily the case that \( S \) controls \( R \). This property of ‘indirect control’ is captured by the following definition.

Definition 2.2. The mutual control structure \( C \) is invariant if it satisfies the following condition.

Indirect control: For all \( S,T,R \in P(N) \) with \( T \subseteq C(S) \) and \( R \subseteq C(S \cup T) \) we have \( R \subseteq C(S) \).

The set of all invariant mutual control structures is denoted by \( C^* \).

The term ‘invariant’ reflects the fact that such a mutual control structure does not change if we add further control relations in the sense of the indirect control property: if \( T \) is controlled by \( S \) and \( R \) is controlled by \( S \) and \( T \) jointly, then \( R \) is already controlled directly by \( S \) and, thus, adding this control relation does not change the mcs.

An invariant mutual control structure \( C \) is transitive: for all \( S,T,R \in P(N) \) with \( T \subseteq C(S) \) and \( R \subseteq C(T) \) we have \( R \subseteq C(S \cup T) \). This follows since \( R \subseteq C(T) \) implies \( R \subseteq C(S \cup T) \) by monotonicity of \( C \).

Examples of (invariant) mutual control structures can be derived from simple games. A simple game is a map \( v : P(N) \to \{0,1\} \) with \( v(\emptyset) = 0 \) and with \( v(S) \leq v(T) \) for all \( S,T \in P(N) \) with \( S \subseteq T \). \(^1\) If \( v(S) = 1 \) then coalition \( S \) is winning, otherwise it is losing. The set of all simple games is denoted by \( \Sigma \).

Example 2.3. Let \( v \in \Sigma \) and define \( C : P(N) \to P(N) \) by

\[
C(S) = \begin{cases} 
N & \text{if } v(S) = 1 \\
\emptyset & \text{if } v(S) = 0
\end{cases}
\]

for all \( S \in P(N) \). Then \( C(\emptyset) = \emptyset \), and \( C(S) \subseteq C(T) \) for all \( S,T \in P(N) \) with \( S \subseteq T \), so \( C \in C \). Also, if \( S,T,R \in P(N) \) with \( T \subseteq C(S) \) and \( R \subseteq C(S \cup T) \) then either \( T = \emptyset \) in which case \( R \subseteq C(S) \) or \( T \neq \emptyset \) in which case \( C(S) = N \) and thus \( R \subseteq C(S) \) as well. Hence, \( C \) satisfies indirect control, so that \( C \in C^* \).

\(^1\)Thus, in our paper, a simple game – like a mutual control structure – is monotonic by definition. Also observe that \( v(N) = 0 \) is allowed.
The remainder of this section is devoted to the question how an arbitrary mutual control structure (mcs) can be turned into an invariant mutual control structure (imcs). This requires, in particular, that indirect control relations are incorporated explicitly into the mcs. Let $C$ be an arbitrary mcs. We define $C^1, C^2, \ldots \in C$ recursively by

$$C^k(S) = \begin{cases} C(S) & \text{if } k = 1 \\ C(C^{k-1}(S) \cup S) & \text{if } k > 1 \end{cases}$$

for each $S \in P(N)$. Observe that in each step of this algorithm the new coalition controlled by $S$ is the coalition controlled by $S$ jointly with the coalition already controlled by $S$ according to the previous step. Thus, this definition is very much in the spirit of the definition of indirect control. Clearly, by monotonicity of $C$, there must be a natural number $p \geq 1$ such that for each $S \in P(N)$ we have

$$C(S) = C^1(S) \subseteq C^2(S) \subseteq \cdots \subseteq C^p(S) = C^{p+1}(S) = C^{p+2}(S) = \cdots \quad (1)$$

Denote $C^p$ by $C^*$. Then we have the following result, which says that in $C^*$ adding players controlled by a coalition $S$ to $S$ does not enlarge the set of players controlled by $S$.

**Lemma 2.4.** Let $C \in \mathcal{C}$. Then for each $S \in P(N)$ we have $C^*(S \cup C^*(S)) = C^*(S)$.

**Proof.** Let $p \geq 1$ be as in the definition of $C^*$ (i.e., as in (1)) and let $S \in P(N)$. Then we have

$$C(S \cup C^*(S)) = C(S \cup C^p(S)) = C^{p+1}(S) = C^*(S)$$

hence

$$C^2(S \cup C^*(S)) = C((S \cup C^*(S)) \cup C(S \cup C^*(S))) = C(S \cup C^*(S)) = C^*(S).$$

By repeating this argument we find

$$C^*(S \cup C^*(S)) = C^p(S \cup C^*(S)) = C^*(S).$$

\[\square\]

A few further observations are collected in the following lemma. We omit the straightforward induction proofs. For $C, D \in \mathcal{C}$, we write $C \subseteq D$ if $C(S) \subseteq D(S)$ for all $S \in P(N)$.

**Lemma 2.5.** Let $C \in \mathcal{C}$. Then $C^k \in \mathcal{C}$ for each $k \geq 1$. If $D \in \mathcal{C}$ with $C \subseteq D$, then $C^k \subseteq D^k$ for each $k \geq 2$.

We now show that $C^*$ is an invariant mcs.

**Proposition 2.6.** Let $C \in \mathcal{C}$. Then $C^* \in \mathcal{C}$.
Proof. In view of Lemma 2.5 we only still have to show indirect control of $C^*$.
Let $S \in P(N)$ and $T \subseteq C^*(S)$. Then by monotonicity of $C^*$ and Lemma 2.4
we have
$$C^*(S) \subseteq C^*(S \cup T) \subseteq C^*(S \cup C^*(S)) = C^*(S),$$
so that $C^*(S \cup T) = C^*(S)$. Hence, if $R \subseteq C^*(S \cup T)$ then $R \subseteq C^*(S)$, so that
indirect control holds.

The following example shows that different mutual control structures may
result in the same invariant mcs.

Example 2.7. Let $N = \{1, \ldots, 4\}$ and let $C, D \in C$ be given by
$$C(\emptyset) = \emptyset, C(\{3\}) = \{3\}, C(\{4\}) = \{4\},$$
$$C(\{3, 4\}) = \{2, 3, 4\}, C(\{1, 2\}) = \{1\}, C(\{2, 3\}) = \{1, 3\}$$
and
$$D(\emptyset) = \emptyset, D(\{3\}) = \{3\}, D(\{4\}) = \{4\},$$
$$D(\{3, 4\}) = N, D(\{1, 2\}) = \{1\}, D(\{2, 3\}) = \{1, 3\}$$
and further satisfying monotonicity. Then it is straightforward to compute that
$C^* = D^* = D$.

Remark 2.8. We define $\cup$ and $\cap$ on $C$ by
$$(C \cup D)(S) = C(S) \cup D(S),$$
$$(C \cap D)(S) = C(S) \cap D(S),$$
for all $S \in P(N)$. Clearly, $C \cup D, C \cap D \in C$ for all $C, D \in C$. Let $C, D \in C^*$
and consider $C \cap D$. If $T \subseteq (C \cap D)(S)$ and $R \subseteq (C \cap D)(S \cup T)$, then
$R \subseteq C(S)$ and $R \subseteq D(S)$ by indirect control. Hence, $R \subseteq (C \cap D)(S)$. Thus, $C \cap D \in C^*$.

However, $C \cup D$ does not have to be invariant. Consider the following mutual
control structures:
$$C(S) = \{\{2\} \text{ if } 1 \in S, \emptyset \text{ otherwise}\}$$
and
$$D(S) = \{\{3\} \text{ if } 2 \in S, \emptyset \text{ otherwise}\}.$$ 
Then $(C \cup D)(\{1\}) = \{2\}$, $[2, 3] = (C \cup D)(\{1, 2\})$, but $[2, 3] \not\subseteq (C \cup D)(\{1\})$.
So $C \cup D$ is not invariant.

In the next section we consider the representation of a mutual control structure
as a vector of simple games.

3 Mutual control structures and simple games

Let $C$ be an mcs. Instead of writing for each coalition all players it controls, we
may equivalently write for each player all coalitions by which it is controlled.
Formally, for each $i \in N$ we define a simple game $w_i^C$ by
$$w_i^C(S) = \begin{cases} 1 & \text{if } i \in C(S) \\ 0 & \text{if } i \not\in C(S). \end{cases}$$
This way with every $C \in \mathcal{C}$ a vector of simple games $w^C \in \Sigma^N$ is associated. Conversely, for a vector of simple games $w \in \Sigma^N$ we can define an mcs $C^w$ by $i \in C^w(S) :\iff w_i(S) = 1$ for all $i \in N$ and $S \in P(N)$. Clearly, these definitions determine a bijection between $\mathcal{C}$ and $\Sigma^N$, and all statements and results for $\mathcal{C}$ can be equivalently formulated for $\Sigma^N$ and conversely.

For an arbitrary $w \in \Sigma^N$ it is not necessarily the case that $C^w$ is invariant, i.e., $C^w \in \mathcal{C}$. For each pair $i, j \in N$ with $i \neq j$ we now define a map $t_{i,j} : \Sigma^N \rightarrow \Sigma^N$, such that repeated application of these maps will transform a $w \in \Sigma^N$ into a $w^* \in \Sigma^N$ with the property that $C^{w^*}$ is invariant. Formally,

$$
t_{i,j}(w_k)(S) = \begin{cases} 
    w_k(S), & \text{if } k \neq i \\
    \tilde{w}_i(S), & \text{if } k = i 
\end{cases}
$$

for each $k \in N$, where

$$
\tilde{w}_i(S) = \begin{cases} 
    1, & \text{if } w_i(S) = 1 \\
    1, & \text{if } S = (S_1 \setminus \{j\}) \cup S_2 \text{ for some } S_1, S_2 \text{ with } w_i(S_1) = 1, w_j(S_2) = 1 \\
    0, & \text{otherwise}
\end{cases}
$$

for all $S \in P(N)$. We call $t_{i,j}(w)$ an elementary substitution. In $t_{i,j}(w)$, the set of winning coalitions in game $w_i$ is extended by replacing player $j$ in every winning coalition of $w_i$ by any winning coalition of $w_j$; i.e., $j$ is replaced by a coalition that controls $j$.\footnote{This coalition may again contain player $j$ but that case is also covered by monotonicity.} This is, clearly, in the spirit of indirect control.

Elementary substitutions are closely related to the procedure constructing invariant mcs discussed in the preceding section. Clearly, since applying elementary substitutions on a vector of simple games $w$ can only increase the collections of winning coalitions, after finitely many steps we must obtain some $w^* \in \Sigma^N$ which is invariant under such substitutions. We show that the associated mcs $C^{w^*}$ is invariant and equal to the imcs $(C^w)^r$ obtained by applying the algorithm of the preceding section to $C^w$. We start with the following lemma.

**Lemma 3.1.** Let $w \in \Sigma^N$ and $i, j \in N$ with $i \neq j$. Then, for each $r \geq 1$, we have

\begin{itemize}
    \item[(a)] $(C^w)^r \subseteq (C^{t_{i,j}(w)})^r$.
    \item[(b)] $(C^{t_{i,j}(w)})^r \subseteq (C^w)^{2r}$.
\end{itemize}

**Proof.** Write $u = t_{i,j}(w)$. For (a), since $u(S) \geq w(S)$ for all $S \in P(N)$, we have $C^u(S) \subseteq C^w(S)$ for all $S$, and the proof is complete by Lemma 2.5.

In order to prove (b), let $S \in P(N)$. The proof is by induction.

First suppose $r = 1$. Let $k \in C^u(S)$, i.e., $u_k(S) = 1$. If $k \neq i$ then $w_k(S) = u_k(S) = 1$, hence $k \in C^w(S) = (C^w)^1(S) \subseteq (C^w)^2(S)$. This argument also holds if $k = i$ and $w_k(S) = 1$. Now suppose $k = i$ and $w_k(S) = 0$. Then there are $S_1, S_2 \in P(N)$ with $w_k(S_1) = 1, w_j(S_2) = 1$, and such that $S = (S_1 \setminus \{j\}) \cup S_2$. Hence $w_j(S) \geq w_j(S_2) = 1$ so that $j \in C^w(S) = (C^w)^1(S)$.
and $w_k(S \cup (C^w)^1(S)) \geq w_k(S \cup \{j\}) \geq w_k(S_1) = 1$. Hence $k \in (C^w)^2(S)$ also in this case.

Now let $r \geq 2$ and suppose (b) holds for all $k \leq r - 1$. Then

$$\begin{align*}
(C^w)^r(S) &= C^u(S \cup (C^w)^{r-1}(S)) \\
&\subseteq (C^w)^2(S \cup (C^w)^{r-1}(S)) \\
&\subseteq (C^w)^2(S \cup (C^w)^{2r-2}(S)) \\
&= C^u(S \cup (C^w)^{2r-2}(S) \cup C^u(S \cup (C^w)^{2r-2}(S))) \\
&= C^u(S \cup (C^w)^{2r-2}(S) \cup (C^w)^{2r-1}(S)) \\
&= C^u(S \cup (C^w)^{2r-1}(S)) \\
&= (C^u)^2(S),
\end{align*}$$

by using the induction hypothesis and Lemma 2.5.

An consequence of this lemma is that, if we first apply an elementary substitution to a vector of simple games, then the associated invariant mcs does not change.

**Corollary 3.2.** Let $w \in \Sigma^N$ and let $i, j \in N$ with $i \neq j$. Then $(C^w)^* = (C^{t_{i,j}(w)})^*$.

**Proof.** Write $u = t_{i,j}(w)$ and take $p \in \mathbb{N}$ such that $(C^w)^r = (C^u)^*$ and $(C^u)^r = (C^u)^*$ for all $r \geq p$. Let $S \in P(N)$. Then

$$(C^w)^*(S) = (C^u)^p(S) \subseteq (C^u)^p(S) \subseteq (C^u)^{2p}(S) = (C^u)^*(S),$$

where the two inclusions follow from Lemma 3.1. Hence

$$(C^w)^*(S) = (C^u)^p(S) = (C^u)^*(S).$$

As already observed, repeatedly applying elementary substitutions to a $w \in \Sigma^N$ must result in some $w^* \in \Sigma^N$ which is invariant under further elementary substitutions, i.e., $t_{i,j}(w^*) = w^*$ for all $i, j \in N$ with $i \neq j$. This is so because by an elementary substitution the set of winning coalitions in each coordinate game can only expand. Let $T(w) \subseteq \Sigma^N$ denote the set of all such $w^*$, i.e., obtainable from $w$ by elementary substitutions and invariant under further elementary substitutions. Below (Corollary 3.5) we will actually show that $T(w)$ contains a unique element for every $w \in \Sigma^N$.

In general, call $\bar{w} \in \Sigma^N$ an extension of $w \in \Sigma^N$ if $\bar{w}_i(S) \geq w_i(S)$ for every $i \in N$ and $S \in P(N)$. Call $w \in \Sigma^N$ invariant if $t_{i,j}(w) = w$ for all $i, j \in N$ with $i \neq j$. Call $\bar{w} \in \Sigma^N$ a minimal invariant extension of $w \in \Sigma^N$ if $\bar{w}$ is an invariant extension of $w$ and for every invariant extension $w' \in \Sigma^N$ of $w$ we have $w'_i(S) \geq \bar{w}_i(S)$ for all $i \in N$ and $S \in P(N)$. Observe that, if a minimal invariant extension of $w$ exists, then it is unique.

We first show that, if $\bar{w}$ is an invariant extension of $w$, then it is an invariant extension of $w^*$ for every $w^* \in T(w)$.
Lemma 3.3. Let \( \bar{w} \) be an invariant extension of \( w \in \Sigma^N \) and let \( w^* \in T(w) \). Then \( \bar{w} \) is an invariant extension of \( w^* \).

**Proof.** Let \( i, j \in N \). It is sufficient to prove that \( \bar{w} \) is an extension of \( t_{i,j}(w) \). Suppose not, then by definition of \( t_{i,j} \) there must be an \( S_1 \in P(N) \) with \( w_i(S_1) = 1 \), a \( j \in S_1 \), and an \( S_2 \in P(N) \) with \( w_j(S_2) = 1 \), such that \( w_i((S_1 \setminus \{j\}) \cup S_2) = 0 \). But then, \( t_{i,j}(\bar{w})((S_1 \setminus \{j\}) \cup S_2) = 1 \) since \( \bar{w} \) is an extension of \( w \), contradicting invariance of \( \bar{w} \).

As announced, in what follows we will show that \( T(w) \) contains a unique element, which is also the unique minimal invariant extension of \( w \). The next lemma further prepares for this result.

Lemma 3.4. Let \( w \in \Sigma^N \). Then \( w \) is invariant if and only if \( C^w = (C^w)^* \).

**Proof.** For the only-if part, let \( t_{i,j}(w) = w \) for all \( i, j \in N \) with \( i \neq j \). It is sufficient to show that \((C^w)^r \subseteq C^w \), hence \((C^w)^r = C^w \), for all \( r \geq 1 \). We show this by induction over \( r \). For \( r = 1 \) this is clear. Now let the claim be true for \( r - 1 \). Let \( S \in P(N) \) and \( k \in (C^w)^r(S) \). Then by the induction hypothesis

\[
k \in (C^w)^r(S) = C^w (S \cup (C^w)^{r-1}(S)) = C^w (S \cup C^w(S)) .
\]

Hence there is a coalition \( T \subseteq C^w(S) \setminus S \) such that \( w_k(S \cup T) = 1 \). If \( T = \emptyset \) then \( w_k(S) = 1 \), hence \( k \in C^w(S) \) and we are done. So let \( T \neq \emptyset \). Then \( w_k(S) = 1 \) for all \( \ell \in T \). We show that \( w_k(S) = 1 \) and, hence, \( k \in C^w(S) \), by induction over \( |T| \). Let \( |T| = 1 \) and \( \{\ell\} = T \). Then

\[
w_k(S) = w_k(S \cup T \setminus \{\ell\}) = w_k((S \cup T \setminus \{\ell\}) \cup S)
= t_{k,\ell}(w)_k(S \cup T) = w_k(S \cup T) = 1,
\]

where the third equality follows from the definition of \( t_{k,\ell} \) and the fourth from invariance of \( w \). Hence \( k \in C^w(S) \). Let now \( m = |T| \geq 2 \) and let the claim be true for \( m - 1 \). This implies in particular that \( w_k(S) = w_k(S \cup P) \) whenever \(|P| = m - 1 \). Then let \( \ell \in T \) and write \( T = T' \cup \{\ell\} \), where \(|T'| = m - 1 \). Then

\[
w_k(S) = w_k(S \cup T') = w_k(S \cup T \setminus \{\ell\}) = w_k((S \cup T \setminus \{\ell\}) \cup S)
= t_{k,\ell}(w)_k(S \cup T) = w_k(S \cup T) = 1,
\]

where the first equality follows by induction, the third equality again from the definition of \( t_{k,\ell} \), and the fourth from invariance of \( w \). Hence, \( k \in C^w(S) \).

For the if-part, let \((C^w)^* = C^w \). Assume that there are \( i, j \in N \) with \( i \neq j \) and \( T \in P(N) \) such that \( t_{i,j}(w)_k(T) \neq w_k(T) \). Then we must have \( i = k \) and \( t_{i,j}(w)_k(T) = 1 \) whereas \( w_k(T) = 0 \). Hence, there are coalitions \( T_1 \) and \( T_2 \) with \( T = (T_1 \setminus \{j\}) \cup T_2 \) such that \( w_k(T_1) = w_j(T_2) = 1 \), \( j \in T_1 \), and \( w_k(T_1 \setminus \{j\}) = 0 \). Hence \( w_j(T) = 1 \), i.e., \( j \in C^w(T) \), and thus \( w_k(T \cup C^w(T)) = 1 \), i.e., \( k \in C^w(T \cup C^w(T)) = (C^w)^2(T) \). On the other hand, \( w_k(T) = 0 \) implies \( k \notin C^w(T) \), a contradiction since \( C^w(T) = (C^w)^2(T) \), as follows from \((C^w)^* = C^w \).

The preceding results now imply existence of (unique) minimal invariant extensions.
Corollary 3.5. Let $w \in \Sigma^N$. Then $|T(w)| = 1$. If $T(w) = \{w^*\}$, then

(a) $C^w = (C^w)^*$, and

(b) $w^*$ is the unique minimal invariant extension of $w$.

Proof. First, let $w^* \in T(w)$. We prove that $w^*$ satisfies (a). Therefore, let $S \in P(N)$. Then $C^w(S) = (C^w)^*(S)$ by Lemma 3.4, and $(C^w)^*(S) = (C^w)^*(S)$ by Corollary 3.2, so that $C^w(S) = (C^w)^*(S)$. This completes the proof of (a).

Now suppose $w^*, w' \in T(w)$. Then by (a), $C^{w^*} = C^{w'}$, so that $w^* = w'$. Hence $|T(w)| = 1$, say $T(w) = \{w^*\}$. Let $\bar{w}$ be any invariant extension of $w$, then by Lemma 3.3, $\bar{w}$ is an invariant extension of $w^*$. This proves (b).

Similarly to vectors of monotonic simple games, also mutual control structures can be extended. We call $D \in C^*$ an invariant extension of $C \in \mathcal{C}$ if $C \subseteq D$. Clearly, $C^*$ is an invariant extension of $C$, but also $D \in C^*$ defined by $D(S) = N$ for all $S \in P(N)$ is an invariant extension of any mutual control structure. Call an invariant extension $D$ of $C$ minimal if $D(S) \subseteq D'(S)$ for every $S \in P(N)$ and every invariant extension $D'$ of $C$. Clearly again, minimal invariant extensions are unique. Then we have the following result.

Proposition 3.6. Let $C \in \mathcal{C}$. Then $C^*$ is the (unique) minimal invariant extension of $C$.

Proof. Clearly, $C^*$ is an invariant extension of $C$. Let $D \in C^*$ be an invariant extension of $C$ and let $S \in P(N)$. It is sufficient to prove that $C^r(S) \subseteq D(S)$ for all $r \geq 1$. For $r = 1$ this follows by definition of an invariant extension. Suppose the claim is true for $r - 1$. Then $C^{r-1}(S) \subseteq D(S \cup C^{r-1}(S))$ again by definition of an invariant extension. By the induction hypothesis, $C^{r-1}(S) \subseteq D(S)$. Hence by indirect control of $D$ we obtain $C^r(S) \subseteq D(S)$.

The main results so far are summarized in the following commuting diagram, where $(\Sigma^N)^* := \{w \in \Sigma^N \mid w \text{ is invariant}\}$.

\[
\begin{array}{ccc}
w \in \Sigma^N & \longrightarrow & C^w \in \mathcal{C} \\
\downarrow & & \downarrow \\
w^* \in (\Sigma^N)^* & \longrightarrow & C^{w^*} = (C^w)^* \in \mathcal{C}^*
\end{array}
\]

4 Power indices for invariant mutual control structures

In this section we develop a class of power indices for invariant mutual control structures. As before, the generic player set is denoted by $N = \{1, \ldots, n\}$, and $\mathcal{C}^*$ is the set of all imcs with this player set. A power index is a map $\varphi : \mathcal{C}^* \rightarrow \mathbb{R}^N$.

For an imcs $C$ the marginal contribution of player $i \in N$ to a coalition $S \subseteq N$ is defined as $\Delta^C_i(S) = C(S) \setminus C(S \setminus \{i\})$. We say that $i \in N$ is a null player (with respect to $C$) if $\Delta^C_i(S) = \emptyset$ and $i \notin C(S)$ for all $S \subseteq N$. That is, player $i$
is a null player if \( i \) is never needed by any coalition to exercise its control, and \( i \) is also not controlled by any coalition. The imcs in which every player is a null player, is denoted by \( O \), i.e., \( O(S) = \emptyset \) for all \( S \subseteq N \).

Let \( \pi : N \to N \) be a permutation. Then we define \( \pi C \in \mathcal{C}^* \) by

\[
(\pi C)(S) = \pi \left( C \left( \pi^{-1}(S) \right) \right).
\]

The basic axioms that we impose on a power index, are stated next.

**Null-Player (NP)** \( \varphi_i(C) = 0 \) for every null player \( i \) with respect to \( C \), for every \( C \in \mathcal{C}^* \).

**Normalization (NO)** \( \sum_{i \in N} \varphi_i(C) = 0 \) for every \( C \in \mathcal{C}^* \).

**Anonymity (AN)** \( \varphi_{\pi(i)}(\pi C) = \varphi_i(C) \) for every player \( i \in N \), every permutation \( \pi \) of \( N \), and every \( C \in \mathcal{C}^* \).

**Transfer Property (TP)** \( \varphi(C) - \varphi(C') = \varphi(D) - \varphi(D') \) for all \( C, C', D, D' \in \mathcal{C}^* \) such that \( C' \subseteq C, D' \subseteq D \), and \( C(S) \setminus C'(S) = D(S) \setminus D'(S) \) for every \( S \subseteq N \).

The normalization axiom does what its name suggests: it provides a normalization by setting the sum of the players’ powers equal to 0. The null-player axiom sets the power of a player who neither controls nor is controlled, equal to 0. We have in mind that a positive power assignment to a player should indicate a positive net controlling power, and a negative assignment the opposite, although this is not yet captured by the above axioms. The anonymity axiom needs no further explanation. The transfer property says that if going from \( C' \) to \( C \) involves exactly the same increase in control as going from \( D' \) to \( D \), then the power of each player should change by the same amount when going from \( C' \) to \( C \) as when going from \( D' \) to \( D \). The transfer property is related to a property with the same name, used to characterize the Shapley value (Shapley, 1953) for (monotonic) simple games (Dubey, 1975). The form in which we present it is closely related to a version of the axiom discussed in Dubey et al (2005). TP is equivalent to a condition closely related to the original format of the transfer axiom as introduced in Dubey (1975).

**Lemma 4.1.** Let \( \varphi \) be a power index. Then \( \varphi \) satisfies TP if and only if

\[
\varphi(C \cap D) + \varphi(C \cup D) = \varphi(C) + \varphi(D)
\]

for all \( C, D \in \mathcal{C}^* \) with \( C \cup D \in \mathcal{C}^* \).

**Proof.** First, let \( \varphi \) satisfy TP and let \( C, D \in \mathcal{C}^* \) with \( C \cup D \in \mathcal{C}^* \). Clearly,

\[
(C(S) \cup D(S)) \setminus C(S) = D(S) \setminus (C(S) \cap D(S))
\]

Note that \( C \cap D \in \mathcal{C}^* \) by Remark 2.8.
for all $S \subseteq N$. Hence by TP, $\varphi(C \cup D) - \varphi(C) = \varphi(D) - \varphi(C \cap D)$, implying (2).

Next, let $\varphi$ satisfy (2) for all $C, D \in C^*$ with $C \cup D \in C^*$. We show that $\varphi$ satisfies TP. Let $C, D \in C^*$ such that $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for all $S \subseteq N$ and define

$$E(S) = \bigcup_{T \subseteq S} C(T) \setminus C'(T) = \bigcup_{T \subseteq S} D(T) \setminus D'(T).$$

Clearly, $E \in C$. Let $p \geq 1$ such that $E^* = E^p$. If $p = 1$ then $E = E^* \in C^*$. Suppose $p \geq 2$. If $i \in E^*(S)$ then there are coalitions $T_1, \ldots, T_{p-1}$ such that

$$T_1 \subseteq E(S),$$

$$T_k \subseteq E\left(S \cup \bigcup_{i=1}^{k-1} T_i\right) \text{ for } k = 1, \ldots, p-1, \quad (3)$$

$$i \in E\left(S \cup \bigcup_{k=1}^{p-1} T_k\right).$$

In particular, we have $T_k \subseteq C(S), T_k \subseteq C\left(S \cup \bigcup_{i=1}^{k-1} T_i\right)$ for $k = 1, \ldots, p-1$, and $i \in C\left(S \cup \bigcup_{k=1}^{p-1} T_k\right)$. As $C$ is invariant, we have $i \in C(S)$, hence $E^*(S) \subseteq C(S)$ for all $S \subseteq N$. Hence, $C \supseteq C' \cup E^*$. Also, if $j \in C(S)$ and $j \notin C'(S)$, then $j \in E(S) \subseteq E^*(S)$, so that $C \subseteq C' \cup E^*$. Thus, $C = C' \cup E^*$. We show that

$$C'(S) \cap E^*(S) = D'(S) \cap E^*(S) \quad (4)$$

for all $S \subseteq N$. For this purpose let $i \in C'(S) \cap E^*(S)$. Then there are $T_1, \ldots, T_{p-1}$ as in (3). By definition of $E$ there is $R \subseteq S \cup \bigcup_{k=1}^{p-1} T_k$ such that $i \in C(R) \setminus C'(R) = D(R) \setminus D'(R)$. In particular, $i \in D(R)$. We also have $T_1 \subseteq E(S) \subseteq D(S)$ and $T_k \subseteq E\left(S \cup \bigcup_{i=1}^{k-1} T_i\right) \subseteq D\left(S \cup \bigcup_{i=1}^{k-1} T_i\right)$ so that $i \in D(S)$ by invariance of $D$. Finally, we have $i \in C'(S)$ and therefore $i \notin C(S) \setminus C'(S) = D(S) \setminus D'(S)$. Hence, $i \in D'(S)$. Thus, $C'(S) \cap E^*(S) \subseteq D'(S) \cap E^*(S)$. The converse inclusion is analogous, so (4) holds. As $C' \cap E^*$ and $D' \cap E^*$ are invariant by Remark 2.8, we find with (2) that

$$\varphi(C) - \varphi(C') = \varphi(C' \cup E^*) - \varphi(C') = \varphi(E^*) - \varphi(C' \cap E^*) = \varphi(E^*) - \varphi(D' \cap E^*) = \varphi(D' \cup E^*) - \varphi(D') = \varphi(D) - \varphi(D').$$

\[\blacksquare\]
Our main theorem will be a characterization of all power indices satisfying NP, NO, AN, and TP. A crucial result for this is the proposition below, which expresses the power index values assigned to an imcs $C$ as sums of values assigned to so-called unanimity imcs. For every $S \subseteq N$ and $j \in N$ we define the *unanimity* mutual control structure $U_{S, \{j\}}$ by

$$U_{S, \{j\}}(T) = \begin{cases} \{j\} & \text{if } S \subseteq T \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to check that $U_{S, \{j\}}$ is an imcs. Similarly, it is easy to check that $\bigcap_{S \in M} U_{S, \{j\}} \in C^*$ for any subset $M \subseteq P(N)$ (see also Remark 2.8).

Let $C \in C^*$. For a coalition $S \subseteq N$ we define the *excess* of $S$ by

$$E^C(S) = C(S) \setminus \bigcup_{S' \subsetneq S} C(S').$$

Coalition $S$ is *minimal controlling* in $C$ if $E^C(S) \neq \emptyset$. Note that $i \in E^C(S)$ if and only if $S$ is a minimal winning coalition in $w^C_i$ (i.e., $S$ is winning and all proper subsets of $S$ are losing). Let $M(C)$ denote the set of minimal controlling coalitions in $C$.

The announced proposition is the following.

**Proposition 4.2.** Let $\varphi$ be a power index satisfying satisfy TP and NP, and let $C \in C^*$. For every $j \in N$ let $M^j = \{S \in M(C) \mid j \in E^C(S)\}$. Then

$$\varphi(C) = \sum_{j \in N} \sum_{t=1}^{\lfloor M^j \rfloor} (-1)^{t+1} \sum_{M \subseteq M^j, |M| = t} \varphi\left( \bigcap_{S \in M} U_{S, \{j\}} \right).$$

(5)

**Proof.** Let $C \in C^*$. Then one easily checks that

$$C = \bigcup_{S \in M(C)} \bigcup_{j \in E^C(S)} U_{S, \{j\}} = \bigcup_{j \in N} \bigcup_{S \in M} U_{S, \{j\}}.$$

For each $j \in N$ we define the control structures $C^j$ and $C^{-j}$ by

$$C^j = \bigcup_{S \in M^j} U_{S, \{j\}},$$

$$C^{-j} = \bigcup_{k \in N \setminus \{j\}} C^k.$$

It is easy to see that $C^j$ is an invariant mcs. Also $C^{-j}$ is an mcs, and we show that it is invariant, as follows. Let $T \subseteq N$, and suppose $\ell \in N$ with $\ell \notin C^{-j}(T)$. It is sufficient to show that $\ell \notin C^{-j}(T \cup C^{-j}(T))$. If $\ell = j$, then clearly $\ell \notin C^{-j}(T \cup C^{-j}(T))$. Now suppose $\ell \neq j$. Since $\ell \notin C^j(T)$ and $\ell \notin C^{-j}(T)$, we have $\ell \notin C^j(T) \cup C^{-j}(T) = C(T)$. Since $C$ is invariant, this
implies \( \ell \notin C(T \cup C(T)) \). Now
\[
C^{-j}(T \cup C^{-j}(T)) \subseteq C^{-j}(T \cup C^{-j}(T) \cup C^j(T)) \\
\subseteq C^{-j}(T \cup C(T)) \cup C^j(T \cup C(T)) \\
= C(T \cup C(T))
\]
so that \( \ell \notin C^{-j}(T \cup C^{-j}(T)) \).

By TP and Lemma 4.1 we have
\[
\varphi(C) = \varphi(C^{-1}) + \varphi(C^1) - \varphi(C^{-1} \cap C^1) = \varphi(C^{-1}) + \varphi(C^1)
\]
where the last equality follows by NP, noting that \( C^{-1} \cap C^1 = O \). Repeating this argument for \( C^{-1}, C^{-2}, \ldots \), results in
\[
\varphi(C) = \sum_{j \in N} \varphi(C^j).
\]

Let \( j \in N \). It remains to show that
\[
\varphi(C^j) = \sum_{t=1}^{\lfloor |M| \rfloor} (-1)^{t+1} \sum_{M \subseteq M \setminus |M|=t} \varphi\left( \bigcap_{S \in M} U^S(j) \right).
\]

For \( |M^j| = 1 \) there is nothing to show. For \( |M^j| = 2 \), say \( M^j = \{S_1, S_2\} \), we have by TP and Lemma 4.1 that
\[
\varphi(C^j) = \varphi\left( U^{S_1(j)} \cup U^{S_2(j)} \right) \\
= \varphi\left( U^{S_1(j)} \right) + \varphi\left( U^{S_2(j)} \right) - \varphi\left( U^{S_1(j)} \cap U^{S_2(j)} \right)
\]
which results in the desired expression. So suppose \( |M^j| \geq 3 \) and let \( S^* \in M^j \). By induction we have
\[
\varphi\left( \bigcup_{S \in M \setminus \{S^*\}} U^S(j) \cap U^{S^*}(j) \right) = \varphi\left( \bigcup_{S \in M \setminus \{S^*\}} U^{S \cup S^*}(j) \right)
\]
\[
= \sum_{t=1}^{|M^j|-1} (-1)^{t+1} \sum_{M \subseteq M \setminus \{S^*\}, |M|=t} \varphi\left( \bigcap_{S \in M} U^{S \cup S^*}(j) \right)
\]
\[
= \sum_{t=1}^{|M^j|-1} (-1)^{t+1} \sum_{M \subseteq M \setminus \{S^*\}, |M|=t} \varphi\left( \bigcap_{S \in M \cup \{S^*\}} U^S(j) \right). \quad (6)
\]
Now
\[
\varphi(C^j) = \varphi \left( \bigcup_{S \in M' \setminus \{S^*\}} U^S(j) \right) + \varphi \left( U^{S^*}(j) \right)
- \varphi \left( \bigcup_{S \in M' \setminus \{S^*\}} U^S(j) \cap U^{S^*}(j) \right),
\]
\[
= \sum_{t=1}^{\lvert M' \rvert - 1} (-1)^{t+1} \sum_{M \subseteq M' \setminus \{S^*\}, \lvert M \rvert = t} \varphi \left( \bigcap_{S \in M} U^S(j) \right) + \varphi \left( U^{S^*}(j) \right)
- \sum_{t=1}^{\lvert M' \rvert - 1} (-1)^{t+1} \sum_{M \subseteq M' \setminus \{S^*\}, \lvert M \rvert = t} \varphi \left( \bigcap_{S \in M \cup \{S^*\}} U^S(j) \right)
\]
\[
= \sum_{t=1}^{\lvert M' \rvert} (-1)^{t+1} \sum_{M \subseteq M' \setminus \{S^*\}, \lvert M \rvert = t} \varphi \left( \bigcap_{S \in M} U^S(j) \right)
\]
where the first and third equalities follow by TP and Lemma 4.1, and the second by (6) and induction. This completes the proof of the lemma.

Recall that the Shapley value (Shapley, 1953) of a simple game \(w_i\) with player set \(N\) is defined by
\[
Sh_k(w_i) = \sum_{S \subseteq N \setminus \{k\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \left( w_i(S \cup \{k\}) - w_i(S) \right)
\]
for every \(k \in N\). This restriction of the Shapley value to simple games is also called the Shapley-Shubik index (Shapley and Shubik, 1954). Alternatively, as is well-known, the dividends \(d_i(S)\) of a game \(w_i\) can be defined, recursively, by
\[
d_i(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ w_i(S) - \sum_{T \subseteq S \setminus \{i\}} d_i(T) & \text{otherwise} \end{cases}
\]
for all \(S \subseteq N\). Then
\[
Sh_k(w_i) = \sum_{S \subseteq N \setminus \{k\}} \frac{d_i(S)}{|S|}
\]
for every \(k \in N\). For an mcs \(C\) and \(i \in N\), we write \(d_i^C\) for the dividends of \(w_i^C\).

For every weight vector \(\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}\), we define the power index \(\Phi^\omega\) by
\[
\Phi^\omega_i(C) = \sum_{k \in N \setminus \{i\}} \left( \sum_{S \subseteq S, k \not\in S} \frac{d_i^C(S)}{|S|} \alpha_{|S|} + \sum_{S : i \in S, k \in S} \frac{d_i^C(S)}{|S|} \beta_{|S|} \right)
- \sum_{k \in N \setminus \{i\}} \left( \sum_{S \subseteq S, k \not\in S} \frac{d_i^C(S)}{|S|} \alpha_{|S|} + \sum_{S : i \in S, k \in S} \frac{d_i^C(S)}{|S|} \beta_{|S|} \right)
\]
(7)
for all $C \in C^*$ and $i \in N$. The expression in brackets in the first line of (7) says that player $i$ receives a kind of weighted Shapley value in the game $w_\alpha^C$; this expresses the power player $i$ derives from his role in controlling player $k$. The weights depend, both on the size of the coalition of whose dividend player $i$ receives a share, and on whether or not the controlled player $k$ is a member of that coalition. Thus, the first line in (7) represents the total power player $i$ acquires from his role in controlling the other players. In the second line, the total (similarly weighted) power that all other players acquire from controlling player $i$, is subtracted.

The central result of this section is the following theorem, in which the class of power indices of the form $\Phi^\omega$ is characterized.

**Theorem 4.3.** Let $\varphi$ be a power index. Then $\varphi$ satisfies NP, NO, AN, and TP if and only if there is a weight vector $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$ such that $\varphi(C) = \Phi^\omega(C)$ for every $C \in C^*$.

**Proof.** First, let $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$. We show that $\Phi^\omega$ satisfies the four axioms. For AN, this is obvious. When $i$ is a null-player in $C \in C^*$, then $w_\omega^C(S) = 0$ for every $S$, so $d_i^\omega(S) = 0$ for every $S$. Moreover, by using the definition of a dividend it is easy to show that $d_i^\omega(S) = 0$ for every $S$ with $i \in S$. Hence, $\Phi^\omega_i(C) = 0$, which shows NP. NO follows almost directly from the definition of $\Phi^\omega$. Finally, for TP, let $C' \subseteq C$ and $D' \subseteq D$ satisfy $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for every $S$. Then it is easily verified that for every player $j$ and every coalition $S$, we have $d_j^\omega(S) - d_j^{\omega'}(S) = d_j^D(S) - d_j^{D'}(S)$. Hence,

$$\Phi^\omega(C) - \Phi^\omega(C') =$$

$$= \sum_{k \in N \setminus \{i\}} \left( \sum_{S : i \in S, k \notin S} \frac{d_k^C(S) - d_k^{\omega'}(S)}{|S|} \alpha_{|S|} + \sum_{S : i \in S, k \in S} \frac{d_k^C(S) - d_k^{\omega'}(S)}{|S|} \beta_{|S|} \right)$$

$$- \sum_{k \in N \setminus \{i\}} \left( \sum_{S : i \notin S, k \in S} \frac{d_k^C(S) - d_k^{\omega'}(S)}{|S|} \alpha_{|S|} + \sum_{S : i \notin S, k \notin S} \frac{d_k^C(S) - d_k^{\omega'}(S)}{|S|} \beta_{|S|} \right)$$

$$= \sum_{k \in N \setminus \{i\}} \left( \sum_{S : i \in S, k \notin S} \frac{d_k^D(S) - d_k^{D'}(S)}{|S|} \alpha_{|S|} + \sum_{S : i \in S, k \in S} \frac{d_k^D(S) - d_k^{D'}(S)}{|S|} \beta_{|S|} \right)$$

$$- \sum_{k \in N \setminus \{i\}} \left( \sum_{S : i \notin S, k \in S} \frac{d_k^D(S) - d_k^{D'}(S)}{|S|} \alpha_{|S|} + \sum_{S : i \notin S, k \notin S} \frac{d_k^D(S) - d_k^{D'}(S)}{|S|} \beta_{|S|} \right)$$

$$= \Phi^\omega(D) - \Phi^\omega(D')$$

which shows TP.

For the converse, let $\varphi$ be a power index satisfying the four axioms. By NP, NO, and AN, there are numbers $\alpha_{|S|}$ for $|S| = 1, \ldots, n-1$ and $\beta_{|S|}$ for
\[ |S| = 2, \ldots, n \] such that for every \( \emptyset \neq S \subseteq N \) and \( j \in N \) we have

\[
\varphi_i \left( U^{S,(j)} \right) = \begin{cases} 
0 & \text{if } i \notin S \text{ and } i \neq j \\
-\alpha_{i|S|} & \text{if } i = j \text{ and } j \notin S \\
\frac{1}{|S|}\alpha_{i|S|} & \text{if } i \in S \text{ and } j \notin S \\
\frac{1}{|S|-1}\beta_{i|S|} & \text{if } i = j \text{ and } j \in S \\
\frac{1}{|S|}\beta_{i|S|} & \text{if } i \in S \setminus \{j\} \text{ and } j \in S.
\end{cases}
\]  

(8)

By Proposition 4.2 and the first part of the proof it is sufficient to prove that with \( \omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2} \), (8) holds for \( \Phi^\omega \) as well. Let \( w \) be the vector of simple games associated with the unanimity control structure \( U^S \). Then \( d_i(T) = 0 \) for every \( i \neq j \) and every \( T \subseteq N \), whereas \( d_j(T) = 1 \) for \( T = S \) and \( d_j(T) = 0 \) otherwise. It is now straightforward to verify (8) for \( \Phi^\omega \).

The family of power indices in Theorem 4.3 is quite large, since the axioms put no restrictions whatsoever on the values of the weights in \( \omega \). The following very natural condition on a power index \( \varphi \) results in nonnegativity of the weights.

**Monotonicity (MO)** \( \varphi_i(C) \geq \varphi_i(D) \) for all \( C, D \in C^* \) and \( i \in N \) such that (i) \( i \in C(S) \Rightarrow i \in D(S) \) and (ii) \( \Delta^C_i(S) \leq \Delta^D_i(S) \) for all \( S \subseteq N \).

**Corollary 4.4.** Let \( \varphi \) be a power index. Then \( \varphi \) satisfies NP, NO, AN, TP, and MO if and only if there is a weight vector \( \omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}_+ \) such that \( \varphi(C) = \Phi^\omega(C) \) for every \( C \in C^* \).

**Proof.** We leave it to the reader to verify that \( \Phi^\omega \) satisfies MO if \( \omega \in \mathbb{R}^{2n-2}_+ \). Conversely, let \( \omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2} \) and suppose \( \Phi^\omega \) is monotonic. If \( S \subseteq N \), \( i \in S \) and \( j \notin S \), then by MO we have \( \Phi^\omega_i(U^S) \geq \Phi^\omega_i(O) \), hence \( \alpha_i/|S| \geq 0 \). If \( S \subseteq N \), \( i \in S \setminus \{j\} \) and \( j \in S \), then again by MO we have \( \Phi^\omega_i(U^S) \geq \Phi^\omega_i(O) \), hence \( \beta_i/|S| \geq 0 \). Clearly, for \( |S| = 1 \), \( \beta_i = 0 \).

A complete determination of the weights is obtained by replacing MO by the following condition.

**Controlled player (CP)** For all \( C \in C^* \), \( j \in C(N) \), and \( i \in N \setminus C(N) \),

\[
\varphi_j(C) = \begin{cases} 
-1 & \text{if } \Delta^C_i(S) = \emptyset \text{ for all } S \subseteq N \\
\varphi_i(C) - 1 & \text{if } \Delta^C_i(S) = \Delta^C_i(S) \text{ for all } S \subseteq N.
\end{cases}
\]

The controlled player condition says that if \( j \) is a ‘controlled player’, i.e., controlled by at least one coalition and, thus, by \( N \), but does not himself exercise any control, then the power of \( j \) is fixed at \(-1\). Also, if \( i \) is an uncontrolled player, i.e., controlled by no coalition at all, but \( i \) and \( j \) exercise exactly the same control, then their difference in power is fixed at \(-1\), that is, \( i \) gets assigned \(-1\) more than \( j \). Note that, if \( \varphi \) also satisfies NP and \( i \) is a null player, then
the second consequence in CP implies the first. We now have the following corollary. Its proof follows again easily from examining the unanimity mutual control structures of the form \( U^{S,(i)} \), and is left to the reader.

**Corollary 4.5.** Let \( \varphi \) be a power index. Then \( \varphi \) satisfies NP, NO, AN, TP, and CP if and only if \( \varphi(C) = \Phi^\omega(C) \) for every \( C \in \mathcal{C}^* \), where \( \omega = (1, \ldots, 1) \in \mathbb{R}^{2n-2} \).

We conclude with showing independence of the axioms in Theorem 4.3.

**Null-Player** Fix \( \varepsilon > 0 \) and for every nonempty \( S \subseteq N \) and \( j \in N \) define

\[
\varphi_i(U^{S,(j)}) = \begin{cases} 
\varepsilon & \text{if } i \notin S \text{ and } i \neq j \\
-1 & \text{if } i = j \text{ and } j \notin S \\
\frac{1}{|S|} - \frac{|N|-|S|-1}{|S|} \varepsilon & \text{if } i \in S \text{ and } j \notin S \\
\frac{1}{|S|} & \text{if } i = j \text{ and } j \in S \\
\frac{1}{|S|} - \frac{|N|-|S|-1}{|S|} \varepsilon & \text{if } i \in S \setminus \{j\} \text{ and } j \in S.
\end{cases}
\]

Extend \( \varphi \) to \( \mathcal{C}^* \) by using Proposition 4.2. Then \( \varphi \) satisfies NO, AN, and TP, but not NP.

**Normalization** Define the power index \( \varphi \) by \( \varphi_i(C) = |C(|i|)| - w_i^C(N) \) for all \( C \in \mathcal{C}^* \) and \( i \in N \). Then \( \varphi \) satisfies NP and AN. Also, for \( C, D \in \mathcal{C}^* \),

\[
\phi_i(C \cup D) + \phi_i(C \cap D) = |(C(|i|) \cup D(|i|))| - w_i^{C \cup D}(N) + |(C(|i|) \cap D(|i|))| - w_i^{C \cap D}(N)
= |C(|i|) + D(|i|)| - (w_i^C(N) + w_i^D(N))
= \phi_i(C) + \phi_i(D)
\]

for all \( i \in N \), so that \( \varphi \) satisfies TP by Lemma 4.1. Consider \( D \in \mathcal{C}^* \) defined by \( D(S) = N \) for all nonempty \( S \subseteq N \). Then \( \varphi_i(D) = |N| - 1 \) for all \( i \in N \), so that \( \varphi \) does not satisfy NO.

**Anonymity** Let \( N = \{1,2\} \) and define \( \varphi \) by \( \varphi(U^{(1),(1)}) = \varphi(U^{(2),(1)}) = \varphi(U^{(1,2),(1)}) = (0,0) \), \( \varphi(U^{(1,2),(2)}) = -\varphi(U^{(1,2),(1)}) = \varphi(U^{(1),(2)}) = (1,-1) \); and by extending \( \varphi \) to \( \mathcal{C}^* \) using Proposition 4.2. Then \( \varphi \) satisfies NP, NO, and TP, but not AN.

**Transfer property** For a simple game \( v \) and \( i \in N \) let \( \sigma_i(v) = |\{S \subseteq N \setminus \{i\} | v(S \cup \{i\}) - v(S) = 1\}| \) and let

\[
Bz_i(v) = \begin{cases} 
\sigma_i(v)/\sum_{j \in N} \sigma_j(v) & \text{if } \sigma_i(v) \neq 0 \\
0 & \text{if } \sigma_i(v) = 0.
\end{cases}
\]

Thus, \( Bz \) is the normalized Banzhaf value (cf. Banzhaf, 1965; Dubey et al., 2005). Define the power index \( \varphi \) by

\[
\varphi_i(C) = \sum_{j \neq i} Bz_j(w_j^C) - \sum_{j \neq i} Bz_j(w_i^C)
\]

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for all $C \in C^*$ and $i \in N$. It is straightforward to verify that $\varphi$ satisfies NP, NO, and AN. We show that it does not satisfy TP by using Lemma 4.1. For $S, S' \subseteq N$ define $U^{S, S'} \in C^*$ by $U^{S, S'}(T) = S'$ if $S \subseteq T$ and $U^{S, S'}(T) = \emptyset$ otherwise. Now take $N = \{1, 2, 3\}$ and let $C = U^{\{1, 2\}, N}$ and $D = U^{\{1, 3\}, N}$. Then $Bz_i (w^C_i) = \frac{1}{2}$ and $Bz_i (w^D_i) = \frac{1}{2}$ for $i = 1, 2, 3$. Hence, $\varphi_1 (C) = \frac{1}{2}$. Further,

$$C \cup D = U^{\{1, 2\}, N} \cup U^{\{1, 3\}, N},$$

$$C \cap D = U^{N, N}.$$ 

Now $Bz_i (w^{C \cup D}_i) = \frac{3}{5}$ for $i = 1, 2, 3$. Hence, $\varphi_1 (C \cup D) = \frac{4}{5}$. Further, by computation or by NO and AN, we have $\varphi (C \cap D) = 0$. Thus,

$$\varphi_1 (C \cup D) + \varphi_1 (C \cap D) = \frac{4}{5} \neq 1 = \varphi_1 (C) + \varphi_1 (D).$$

So, $\varphi$ does not satisfy TP.

5 Concluding remarks

6 Conclusion

We have introduced a new model to represent mutual control within a set of players. Our model can be understood as a collection of simple games, one for each player. A similar model has been introduced earlier by Gambarelli and Owen (1994) where a set of investors controls a set of firms. However, our model is much more general: Investors in the article of Gambarelli and Owen (1994) are simply uncontrolled players in our model.

Crama and Leruth (2011) point out that the concepts of control and ownership must be carefully distinguished. The following example emphasizes this claim.

Example 6.1. Let $a, b, c, d$ be firms such that $a$ owns 80% of the shares of $b$ and $c$ owns the remaining 20% of $b$. Let further $a$’s shares distributed such that $c$ owns 40% and $d$ owns 60%. Assume that all decisions can be made by an absolute majority. From a financial point of view $c$ owns 32% of $b$ indirectly via $a$ and 20% directly, so all together 52% which is the majority. On the other hand $d$ owns 48% of $b$ indirectly via $a$. However, if we build a mutual control structure $C$ to represent this situation, we find $C (\{a\}) = \{b\}$, $C (\{d\}) = \{a\}$ and $C (\{c\}) = \emptyset$. For the minimal invariant extension of $C$ we have $C^* (\{d\}) = \{a, b\}$ and $C^* (\{c\}) = \emptyset$.

We see that indirect ownership of firms might not lead to any control at all. In particular, while indirect ownership is proportional to the number of shares, indirect control follows a winner takes all principle: Although firm $c$ owns 52% of firm $b$ its indirect votes are useless as it can not force firm $a$ to use it.
In future research, mutual control structures could be used to extend TU games. Myerson (1977) considered TU games together with a graph on the set of players which represents possible cooperation. Particularly, he defined and characterized a value on such structures based on the Shapley value. In the same way we can combine a TU game with a mutual control structure, having in mind that players cannot only cooperate with each other but can also exercise their power over each other. In this case a control structure can severely affect the positions of players within the game.

References


